THE NONSTATIONARY NEAR-SONIC FLOW OF A STREAM AROUND A CORNER

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A flat membrane separates a half space filled with a moving compressible gas from a vacuum. The membrane is broken suddenly, and the resulting plane non-steady-state flow of the gas is investigated in the approximation of near-sonic flow. The flow pattern is constructed using exact particular solutions of the equations of non-steady-state near-sonic flow [1, 2].

§ 1. Let the plane y = 0 be the impenetrable barrier. A gas moves with some velocity $u \le a_*$ along the x axis in the region $y > 0$. At time t = 0 the half-plane $y = 0$, $x > 0$ is suddenly removed. At this moment the region of uniform flow has a direct boundary with the vacuum, and the free surface coincides with the half-plane. Subsequently rarefaction waves spread out from every point on the free surface, and a centered rarefaction wave arises in the neighborhood of the point 0. In this region the flow is governed by the equations of nonstationary near-sonic flow [2–4]

$$\partial U / \partial \tau - U \partial U / \partial X + \partial V / \partial Y = 0,$$
$$\partial V / \partial X + \partial U / \partial Y = 0$$
$$\tau = t / 2t_*, \qquad x = \varepsilon X / a_* t_*, \qquad y = \varepsilon^{1/2} Y / a_* t_*$$
$$u - a_* = \varepsilon U / (\varkappa + 1), \qquad v = \varepsilon^{3/2} V / (\varkappa + 1). \tag{1.1}$$

Here u, v are the components of the velocity vector, a_* is the critical velocity of sound, $\varkappa$ is the adiabatic coefficient, t_* is some characteristic time, ε is a small quantity.

The last two equations impose limitations on flows which may be treated within the framework of the near-sonic approximation. For t = 0 these equations break down on the semi-axis x > 0 since near the point x = 0, y = 0 the stream flows around a finite corner, and the components u and v of the velocity vector change roles as it were: v becomes comparable with a_*, and u becomes small. In the near-sonic approximation it is assumed that $U = \infty$, $V = \infty$ in such region of flow. It is impossible to investigate the dynamics of the free surface in the near-sonic approximation for the problem as formulated here. This breakdown of the equations does not affect the stream because of the supersonic nature of the flow. The region of the stream perturbed by the effect of the free surface is separated from the region where the disturbance has not been reached by the envelope of the rarefaction waves having zero amplitude, which is the characteristic surface of

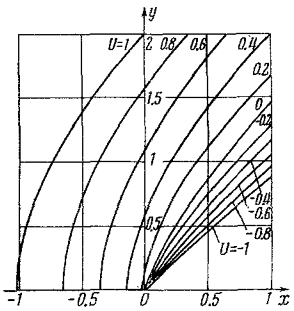

Fig. 1

equations (1.1). The equation for the characteristic surfaces of the system (1.1) has the form

$$\partial X / \partial \tau + U + (\partial X / \partial Y)^2 = 0. \tag{1.2}$$

The required surface propagates with a motion which is uniform but varies in time $u = u_0(\tau)$ or

$$U = U_0(\tau), \; V = U_0'(\tau)\, Y. \tag{1.3}$$

At time t = 0 it should correspond with the half-line Y = 0, X > 0. Integrating (1.2) under these conditions we obtain the equation of this characteristic surface:

$$X = \frac{1}{4\tau} Y^2 - \int_0^\tau U(t)\, dt. \tag{1.4}$$

The flow in the region of the point O for t > 0 is Prandtl-Mayer flow, i.e., from (1.1) for $X \to 0$, $Y \to 0$,

$$U = -X^2 / Y^2, \qquad V = -{}^2/_3 X^3 / Y^3. \tag{1.5}$$

Thus the required solution of Eqs. (1.1) should pass to the flow (1.3) at the characteristic (1.4), and to the flow (1.5) in the neighborhood of the point O.

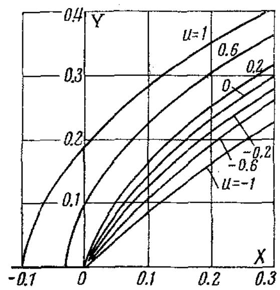

Fig. 2

§ 2. We shall construct the flow pattern in the disturbed region using the exact particular solutions found in papers [1,2]:

$$U = \varphi_2(q, \tau) Y^2 + \varphi_0(q, \tau),$$
$$V = \psi_3(q, \tau) Y^3 + \psi_1(q, \tau) Y, \; X = qY^2 + \chi_0(q, \tau). \tag{2.1}$$

Here φ_2, ψ_1, ψ_3 are determined from the equations

$$\varphi_{2\tau q} - \varphi_{2qq}(\varphi_2 + 4q^2) - \varphi_{2q}(\varphi_{2q} - 2q) - 2\varphi_2 = 0$$
$$\psi_1 = \varphi_{0q}(\varphi_2 + 4q^2) - \varphi_{0\tau},$$
$$\psi_3 = {}^1/_3 [\varphi_{2q}(\varphi_2 + 4q^2) - 4q\varphi_2 - \varphi_{2\tau}], \tag{2.2}$$

and φ_0, χ^0 from the equations

$$\varphi_{0q} = -2q\nu, \qquad \chi_{0q} = \nu,$$
$$\nu_q(\varphi_2 + 4q^2) + 8_q \nu - \nu_\tau = 0. \tag{2.3}$$

The flow pattern (2.1) should pass to that of (1.3) on the characteristic surface (1.4) where $q = q_0(\tau)$. This requirement leads to the following conditions:

$$q = q_0(\tau) = 1/4\tau, \qquad \varphi_2(q_0, \tau) = 0, \qquad \psi_3(q_0, \tau) = 0$$
$$\varphi_0(\tau, q_0) = U_0(\tau), \qquad \chi_0(\tau, q_0) = -\int_0^\tau U(\tau)\, d\tau,$$
$$\psi_1(\tau, q_0) = -U_0'(\tau). \tag{2.4}$$

Condition (1.4) will be satisfied at the point O if

$$\varphi_2 / q^2 \to -1, \qquad \psi_3 / q^3 \to -{}^2/_3,$$
$$\varphi_0 \to 0, \qquad \chi_0 \to 0 \qquad \text{as } q \to \infty. \tag{2.5}$$

Equation (2.2) has the particular solution

$$\varphi_2 = -q^2 + c(\tau) \tag{2.6}$$

satisfying (2.5); $c(\tau)$ is an arbitrary function. From the last two expressions of (2.4) we have

$$c(\tau) = 1/16\ \tau^2.$$

Here

$$\varphi_2 = -q^2 + \frac{1}{16\tau^2}, \qquad \psi_3 = -\frac{2}{3}q^3 - q\frac{1}{8\tau^2} + \frac{1}{24\tau^3}. \tag{2.7}$$

Taking (2.5) and (2.7) into account, Eqs. (2.3) assume the form

$$\varphi_0(q,\tau) = -2\int_\infty^q q\nu\,dq, \qquad \chi_0 = \int_\infty^q \nu\,dq$$

$$\nu_q(3q^2 + 1/16\tau^2) - \nu_\tau + 8q\nu = 0. \tag{2.8}$$

It follows from (2.4) that

$$\int_\infty^{1/4\tau} q\nu\,dq = -\frac{1}{2}U_0(\tau), \qquad \int_\infty^{1/4\tau} \nu\,dq = -\int_0^\tau U_0(\tau)\,d\tau. \tag{2.9}$$

It may easily be shown that when one of these conditions if fulfilled, the other is also fulfilled identically. In fact, differentiating the second expression of (2.9), we obtain

$$\int_\infty^{1/4\tau} \nu_q\left(3q^2 + \frac{1}{16\tau^2}\right)dq + \int_\infty^{1/4\tau} 8q\nu\,dq -$$

$$-\frac{\nu(q_0,\tau)}{4\tau^2} = 2\int_\infty^{1/4\tau} q\nu\,dq = -U_0(\tau).$$

We note that the conditions imposed on Ψ_1 are also fulfilled identically in this case. The solution of Eq. (2.8) has the form

$$\nu = F(\eta)(\sqrt{\tau} - 3\eta)^{2/3}\tau^{2/3}, \qquad \eta = \sqrt{\tau}\,\frac{1-4q\tau}{1-12q\tau}. \tag{2.10}$$

Here $F(\eta)$ is an arbitrary function which may be found from (2.9). Passing from q to η, we obtain

$$\int_0^{1/3\sqrt{\tau}} F(\eta)(1/3\sqrt{\tau} - \eta)^{-1/3}d\eta = \frac{15}{2}U_0(\tau)\tau^{1/3}. \tag{2.11}$$

Therefore

$$F(\eta) = \frac{15\sqrt{3}}{4\pi}\frac{d}{d\eta}\int_0^\eta \frac{U_0(\tau)\tau^{1/3}}{(\eta-\tau)^{2/3}}d\tau. \tag{2.12}$$

We have thus obtained an exact solution satisfying the boundary conditions (2.4) and (2.5).

For $U_0(\tau) = A\tau^\alpha$, ($\alpha \geq 0$) Eq. (2.11) may easily be integrated. The solutions (2.1) assume a particularly simple form in the case $U_0(\tau) = U_0 = \text{const}$. In this case $F = 5 \cdot 3\ 2/3\ U_0$ and the solutions take the form

$$U = -\frac{16q^2\tau^2 - 1}{16\tau^2}Y^2 + \frac{U_0}{\sqrt[3]{2}}\frac{20q\tau - 1}{(12q\tau - 1)^{5/3}},$$

$$X = qy^2 - \frac{U_0\tau}{(6q\tau - 1/2)^{5/3}},$$

$$V = \left(\frac{1}{24\tau^3} - \frac{2}{3}q^3 - q\frac{1}{8\tau^2}\right)Y^3 +$$

$$+ 10\sqrt[3]{2}\,U_0\frac{(4q\tau - 1)q}{(1 - 12q\tau)^{5/3}}Y. \tag{2.13}$$

The solution thus obtained enables us to examine how the process of reaching Prandtl-Meyer behaves in time, and also to examine how the rarefaction waves affect the initial flow. Figures 1 and 2 show the velocity fields for the two moments of time $\tau = 0.1$ and $\tau = 1$, $U_0 = 1$.

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